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DISTORTING MIXED TSIRELSON SPACES*

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ABSTRACT

Any regular mixed Tsirelson space $T(\theta_n, S_n)_{\mathbf{N}}$ for which $\theta_n/\theta^n \to 0$, where $\theta = \lim_n \theta_n^{1/n}$, is shown to be arbitrarily distortable. Certain asymptotic ℓ_1 constants for those and other mixed Tsirelson spaces are calculated. Also, a combinatorial result on the Schreier families $(S_\alpha)_{\alpha<\omega_1}$ is proved and an application is given to show that for every Banach space X with a basis (e_i), the two Δ-spectrums $\Delta(X)$ and $\Delta(X,(e_i))$ coincide.

1. Introduction

A Banach space X with basis (e_i) is asymptotic ℓ_1 if there exists $\delta > 0$ such that for all n and block bases $(x_i)_1^n$ of $(e_i)_n^\infty$,

$$\|\sum_{i=1}^n x_i\| \geq \delta \sum_{i=1}^n \|x_i\|. \tag{1}$$

Such a space need not contain ℓ_1 as witnessed by Tsirelson's famous space T. The complexity of the asymptotic ℓ_1 structure within X can be measured by certain constants $\delta_\alpha(e_i)$ for $\alpha < \omega_1$. $\delta_1(e_i)$ is the largest $\delta > 0$ satisfying (1) above. Subsequent δ_α's are defined by a similar formula where $(x_i)_1^n$ ranges over

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"α-admissible" block bases (all terms are precisely defined in section 2). These notions were developed in [OTW] where, in addition, $\delta_\alpha(y_i)$ was considered, for a block basis (y_i) of (e_i). In this setting, (y_i) becomes the reference frame and one naturally has $\delta_\alpha(y_i) \geq \delta_\alpha(e_i)$. These constants can perhaps increase by passing to further block bases and this leads to the notion of the Δ-spectrum of X, $\Delta(X)$. Roughly, $\Delta(X)$ is the set of all $\gamma = (\gamma_\alpha)_{\alpha<\omega_1}$ where γ_α is the stabilization of $\delta_\alpha(y_i)$ for (y_i) some block basis of (e_i). Alternatively, by keeping (e_i) as the reference frame, in a similar manner we obtain $\Delta(X,(e_i))$. In section 3 we prove that these two notions coincide, $\Delta(X) = \Delta(X,(e_i))$.

Argyros and Deliyanni [AD] constructed the first example of an asymptotic ℓ_1 arbitrarily distortable Banach space by constructing "mixed Tsirelson spaces" and proving that such spaces can be arbitrarily distortable. In section 4 we consider the simplest class of mixed Tsirelson spaces $X = T(\theta_n, S_n)_{n\in\mathbf{N}}$ where $\theta_n \to 0$ and $\sup_n \theta_n < 1$. These are reflexive asymptotic ℓ_1 spaces having a 1-unconditional basis (e_i). Also we may assume $\theta \equiv \lim_n \theta_n^{1/n}$ exists. We prove that if $\theta_n/\theta^n \to 0$ then X is arbitrarily distortable. In particular, this happens if $\theta = 1$. Thus, for example, $T(\frac{1}{n+1}, S_n)_{\mathbf{N}}$ is an arbitrarily distortable space. We also calculate the asymptotic constants $\ddot{\delta}_\alpha(X)$ for these spaces along with the spectral index $I_\Delta(X)$. $\ddot{\delta}_\alpha(X)$ is the supremum of $\delta_\alpha((x_i), |\cdot|)$ under all equivalent norms on X and $I_\Delta(X)$ is the first ordinal α for which $\ddot{\delta}_\alpha(X) < 1$.

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2. Preliminaries

$X, Y, Z, \ldots$ shall denote separable infinite dimensional Banach spaces. *All the spaces we consider will have bases.* Every Banach space with a basis can be viewed as the completion of c_{00} (the linear space of finitely supported real valued sequences) under a certain norm. (e_i) will denote the unit vector basis for c_{00} and whenever a Banach space $(X, \|\cdot\|)$ with a basis is regarded as the completion of $(c_{00}, \|\cdot\|)$, (e_i) will denote this (normalized) basis. If $x \in c_{00}$ and $E \subseteq \mathbf{N}$, $Ex \in c_{00}$ is the restriction of x to E; $Ex(j) = x(j)$ if $j \in E$ and 0 otherwise. Also the **support** of x, $\operatorname{supp}(x)$, (w.r.t. (e_i)) is the set $\{j \in \mathbf{N}\colon x(j) \neq 0\}$. The **range** of x, $\operatorname{ran}(x)$, (w.r.t. (e_i)) is the smallest interval which contains $\operatorname{supp}(x)$. If x, x_1, $x_2, \ldots$ are vectors in X (and $k \in \mathbf{N}$) then we say that x is an average of $(x_i)_i$ (of length k) if there exists $F \subset \mathbf{N}$ with $x = \frac{1}{|F|}\sum_{i\in F} x_i$ (and $|F| = k$). We say that a sequence (y_i) is a **(convex) block sequence of** (x_i) if for all i,

$y_i = \sum_{j=m_i}^{m_{i+1}-1} \alpha_j x_j$ for some sequence $m_1 < m_2 < \cdots$, of integers and $(\alpha_j)_{j\in\mathbf{N}} \subset \mathbf{R}$ (resp. with $\alpha_j \geq 0$ for all j, and $\sum_{j=m_i}^{m_{i+1}-1} \alpha_j = 1$ for all i). If (x_i) is a block basis of (y_i) we write $(x_i) \prec (y_i)$. $X \prec Y$ shall mean that X has a basis which is a block basis of a certain basis for Y, when the given bases are understood. For $\lambda > 1$, $(X, \|\cdot\|)$ is **λ-distortable** if there exists an equivalent norm $|\cdot|$ on X so that for all $Y \prec X$

$$D(Y, |\cdot|) \equiv \sup\left\{\frac{|y|}{|z|} : y, z \in Y, \|y\| = \|z\| = 1\right\} \geq \lambda.$$

X is **distortable** if it is λ-distortable for some $\lambda > 1$ and **arbitrarily distortable** if it is λ-distortable for all $\lambda > 1$. X is of **D-bounded distortion** if for all equivalent norms $|\cdot|$ on X and for all $Z \prec X$ there exists $Y \prec Z$ with $D(Y, |\cdot|) \leq D$. Note that if

$$C = \inf\left\{\frac{\|y\|}{|y|} : y \in Y, y \neq 0\right\}$$

then

$$C\,|y| \leq \|y\| \leq D(Y, |\cdot|)C\,|y| \qquad \text{for all } y \in Y. \tag{2}$$

For more information on distortion we recommend the reader consult the following papers: [S], [MT], [OS1], [OS2], [OS3], [Ma], [T], [OTW].

Asymptotic ℓ_1 Banach spaces are defined by (1) in section 1 (for another approach to asymptotic structure see [MMT]). These spaces were studied in [OTW] where certain asymptotic constants were introduced. We shall recall the relevant definitions but first we need to recall the definition of the Schreier sets S_α, $\alpha < \omega_1$ [AA]. For $F, G \subset \mathbf{N}$, we write $F < G$ when $\max(F) < \min(G)$ or one of them is empty, and we write $n \leq F$ instead of $\{n\} \leq F$. Also for $x, y \in c_{00}$, $x < y$ means $\operatorname{ran}(x) < \operatorname{ran}(y)$.

Definition 2.1: $S_0 = \{\{n\} : n \in \mathbf{N}\} \cup \{\emptyset\}$. If $\alpha < \omega_1$ and S_α has been defined,

$$S_{\alpha+1} = \left\{\bigcup_1^n F_i : n \in \mathbf{N}, n \leq F_1 < F_2 < \cdots < F_n \text{ and } F_i \in S_\alpha \text{ for } 1 \leq i \leq n\right\}.$$

If α is a limit ordinal choose $\alpha_n \nearrow \alpha$ and set

$$S_\alpha = \{F : n \leq F \in S_{\alpha_n} \text{ for some } n\}.$$

If $(E_i)_1^\ell$ is a finite sequence of non-empty subsets of $\mathbf{N}$ and $\alpha < \omega_1$ then we say that $(E_i)_1^\ell$ **is α-admissible** if $E_1 < \cdots < E_\ell$ and $(\min E_i)_1^\ell \in S_\alpha$. If (e_i) is a basic sequence and $(x_i)_1^\ell \prec (e_i)$ then $(x_i)_1^\ell$ is **α-admissible with respect to** (e_i) if $(\text{ran}(x_i))_1^\ell$ is α-admissible where the range of x, $\text{ran}(x)$, is w.r.t. (e_i). If $x \in \text{span}(x_i)$, then x is **α-admissible w.r.t.** (x_i) if $\text{supp}(x)$ (w.r.t. (x_i)) $\in S_\alpha$. Also if $x \in \text{span}(x_i)$ then x **is a 1-admissible average of** (x_i) **w.r.t.** (e_i) if there exists a finite set $F \subset \mathbf{N}$ such that $x = \frac{1}{|F|}\sum_{i\in F} x_i$ and $(x_i)_{i\in F}$ is 1-admissible w.r.t. (e_i). Note that if x is a 1-admissible average of (x_i) w.r.t. (e_i) and for some $\alpha < \omega_1$ each x_i is α-admissible w.r.t. (e_i) then x is $\alpha+1$-admissible w.r.t. (e_i). Thus if (x_i) is a basis for X then X is asymptotic ℓ_1 iff

$$\begin{aligned}0 < \delta_1(x_i) &\equiv \delta_1(X) \equiv \delta_1(X, \|\cdot\|)\\ &= \sup\Big\{\delta \geq 0\colon \|\sum_1^n y_i\| \geq \delta \sum_1^n \|y_i\| \text{ whenever } (y_i)_1^n \prec (x_i)\\ &\qquad \text{and } (y_i)_1^n \text{ is 1-admissible w.r.t. } (x_i)\Big\}.\end{aligned}$$

In [OTW] this definition was extended as follows: For $\alpha < \omega_1$

$$\begin{aligned}\delta_\alpha(x_i) &\equiv \delta_\alpha(X) \equiv \delta_\alpha(X, \|\cdot\|)\\ &= \sup\{\delta \geq 0\colon \|\sum_1^n y_i\| \geq \delta \sum_1^n \|y_i\| \text{ whenever } (y_i)_1^n \prec (x_i)\\ &\qquad \text{and } (y_i)_1^n \text{ is } \alpha\text{-admissible w.r.t. } (x_i)\}.\end{aligned}$$

Observation 2.2: Note that if we have two equivalent norms $\|\cdot\|$, $|\!|\!|\cdot|\!|\!|$ on X and for some $c, C > 0$, $c|\!|\!|x|\!|\!| \leq \|x\| \leq C|\!|\!|x|\!|\!|$ for all $x \in X$, then for all $\alpha < \omega_1$,

$$\frac{c}{C}\delta_\alpha(X, |\!|\!|\cdot|\!|\!|) \leq \delta_\alpha(X, \|\cdot\|) \leq \frac{C}{c}\delta_\alpha(X, |\!|\!|\cdot|\!|\!|).$$

In problems of distortion one is concerned with block bases and equivalent norms. Thus we also consider [OTW]

$$\begin{aligned}\dot{\delta}_\alpha(x_i) &= \sup\{\delta_\alpha(y_i)\colon (y_i) \prec (x_i)\} \quad \text{and}\\ \ddot{\delta}_\alpha(x_i) &= \sup\{\dot{\delta}_\alpha((x_i), |\cdot|)\colon |\cdot| \text{ is an equivalent norm on } X\}.\end{aligned}$$

If $(y_i) \prec (x_i)$ then $\delta_\alpha(y_i) \geq \delta_\alpha(x_i)$. This is because each S_α is **spreading** (if $(n_i)_1^k \in S_\alpha$ and $m_1 < \cdots < m_k$ with $n_i \leq m_i$ for all $i = 1, \ldots, k$, then $(m_i)_1^k \in S_\alpha$). This leads to the following definition [OTW].

Definition 2.3: A basic sequence (y_i) Δ-stabilizes $\gamma = (\gamma_\alpha)_{\alpha<\omega_1} \subseteq \mathbf{R}$ if there exists $\varepsilon_n \searrow 0$ so that for all $\alpha < \omega_1$ there exists $m \in \mathbf{N}$ so that for all $n \geq m$ if $(z_i) \prec (y_i)_n^\infty$ then $|\ \delta_\alpha(z_i) - \gamma_\alpha\ | < \varepsilon_n$.

Remark: It is automatic from the definition that if (y_i) Δ-stabilizes γ then for all $\alpha < \omega_1$, $\gamma_\alpha = \sup\{\delta_\alpha(z_i)\colon (z_i) \prec (y_i)\}$. Furthermore, if $(z_i) \prec (y_i)$ then (z_i) Δ-stabilizes γ.

It is shown in [OTW] that if X has a basis (x_i) and $(y_i) \prec (x_i)$ then there exists $(z_i) \prec (y_i)$ and $\gamma = (\gamma_\alpha)_{\alpha<\omega_1}$ so that (z_i) Δ-stabilizes γ.

Definition 2.4: Let X have a basis (x_i). The Δ-spectrum of X, $\Delta(X)$, is defined to be the set of all γ's so that (y_i) stabilizes γ for some $(y_i) \prec (x_i)$. We also define $\ddot{\Delta}(X) = \bigcup\{\Delta(X, |\cdot|)\colon |\cdot| \text{ is an equivalent norm on } X\}$.

We have that $\Delta(X) \neq \emptyset$ and it is easy to see that $\ddot{\delta}_\alpha(X) = \sup\{\gamma_\alpha\colon \gamma \in \ddot{\Delta}(X)\}$.

THEOREM 2.5 ([OTW]): *Let X have a basis (x_i).*

1. *If $\gamma \in \Delta(X)$ then γ_α is a continuous decreasing function of α. Also $\gamma_{\alpha+\beta} \geq \gamma_\alpha \gamma_\beta$ for all $\alpha, \beta < \omega_1$.*
2. *For all $\alpha < \omega_1$ and $n \in \mathbf{N}$, $\ddot{\delta}_{\alpha\cdot n}(X) = (\ddot{\delta}_\alpha(X))^n$.*
3. *X does not contain ℓ_1 iff $\ddot{\delta}_\alpha(X) = 0$ for some $\alpha < \omega_1$.*

Definition 2.6: Let X have a basis (x_i). The **spectral index** $I_\Delta(X)$ is defined by $I_\Delta(X) = \inf\{\alpha < \omega_1\colon \ddot{\delta}_\alpha(X) < 1\}$ if such an α exists and $I_\Delta(X) = \omega_1$, otherwise.

Definition 2.7 (Mixed Tsirelson Norms [AD]): Let $F \subseteq \mathbf{N}$. Let $(\alpha_n)_{n\in F}$ be a set of countable ordinals and $(\theta_n)_{n\in F} \subset (0,1)$. The mixed Tsirelson space $T(\theta_n, S_{\alpha_n})_{n\in F}$ is the completion of c_{00} under the implicit norm

$$\|x\| = \|x\|_\infty \vee \sup_{q\in\mathbf{N}} \sup\{\theta_q \sum_1^n \|E_i x\|\colon (E_i)_1^n \text{ is an } \alpha_q\text{-admissible sequence of sets}\}.$$

It is proved in [AD] that such a norm exists. They also proved that $T(\theta_n, S_{a_n})_{n\in F}$ is reflexive if F is finite or $\lim_{F\ni n\to\infty} \theta_n = 0$. (e_n) is a 1-unconditional basis for $T(\theta_n, S_{\alpha_n})$ so we can restrict the E_i's in the above definition to be **intervals**. It is worth noting that T, Tsirelson's space [Ts] as described in [FJ], satisfies $T = T(1/2, S_1) = T(1/2^n, S_n)_{\mathbf{N}}$.

3. A property of the Δ-spectrum

The definition of $\delta_\alpha(x_i)$ is w.r.t. the coordinate system (x_i). In [OTW] the following notion is also introduced:

Definition 3.1: Let (e_i) be a basis for X and let $(x_i) \prec (e_i)$. For $\alpha < \omega_1$ we define

$$\delta_\alpha((x_i),(e_i)) = \sup\{\delta \geq 0\colon \|\sum_1^n y_i\| \geq \delta \sum_1^n \|y_i\| \text{ whenever } (y_i)_1^n \prec (x_i) \text{ and } (y_i)_1^n \text{ is } \alpha\text{-admissible w.r.t. } (e_i)\}.$$

If $(y_i) \prec (e_i)$ we say that (y_i) $\Delta_{(e_i)}$-stabilizes $\gamma = (\gamma_\alpha)_{\alpha<\omega_1}$ if there exists $\varepsilon_n \searrow 0$ so that for all $\alpha < \omega_1$ there exists $m \in \mathbf{N}$ so that if $n \geq m$ and $(z_i) \prec (y_i)_n^\infty$ then $|\ \delta_\alpha((z_i),(e_i)) - \gamma_\alpha\ | < \varepsilon_n$. Let $\Delta(X,(e_i))$ be the set of all γ's so that (y_i) $\Delta_{(e_i)}$-stabilizes γ for some $(y_i) \prec (e_i)$.

One can show, by the same arguments used to establish the analogous result for $\Delta(X)$ [OTW], that for all $(x_i) \prec (e_i)$ there exists $(y_i) \prec (x_i)$ and $\gamma = (\gamma_\alpha)_{\alpha<\omega_1}$ so that (y_i) $\Delta_{(e_i)}$-stabilizes γ. In particular, $\Delta(X,(e_i))$ is non-empty.

In this section we prove that the Δ-stabilization and the $\Delta_{(e_i)}$-stabilization are actually the same notions. More precisely we prove

THEOREM 3.2: *Let X have a basis (e_i) and let $(x_i) \prec (e_i)$ so that (x_i) $\Delta_{(e_i)}$-stabilizes $\bar\gamma \in \Delta(X,(e_i))$ and (x_i) Δ-stabilizes $\gamma \in \Delta(X)$. Then $\bar\gamma = \gamma$. Hence $\Delta(X) = \Delta(X,(e_i))$.*

First we need a combinatorial result. $[\mathbf{N}]$ denotes the set of infinite subsequences of $\mathbf{N}$. If $N = (n_i) \in [\mathbf{N}]$ then $S_\alpha(N) = \{(n_i)_{i\in F}\colon F \in S_\alpha\}$ and $[N]$ is the set of infinite subsequences of N.

PROPOSITION 3.3: *Let $N \in [\mathbf{N}]$. Then there exists $L = (\ell_i) \in [N]$ so that for all $\alpha < \omega_1$,*

$$(\ell_i)_{i\in F} \in S_\alpha \Rightarrow (\ell_{i+1})_{i\in F} \in S_\alpha(N).$$

Proof: Let $N = (n_i)$. We shall choose $M = (m_i) \in [N]$ and then prove by induction on α that $L = (\ell_i)$ satisfies the proposition where $\ell_i = n_{m_i}$. Let $m_1 = n_1$. If m_k has been defined set $m_{k+1} = n_{m_k}$.

The case $\alpha = 0$ is trivial.

Assume the result holds for α and that $(n_{m_i})_{i\in F} \in S_{\alpha+1}$. Thus there exists $k \in \mathbf{N}$ and $n_{m_k} \leq E_1 < E_2 < \cdots < E_{n_{m_k}}$ (some possibly empty) so that $E_j \in S_\alpha$ for all j and $(n_{m_i})_{i\in F} = \bigcup_1^{n_{m_k}} E_j$. For each j let $E_j = (n_{m_i})_{i\in F_j}$. Then $n_{n_{m_k}} = n_{m_{k+1}} \leq (n_{m_{i+1}})_{i\in F} = \bigcup_1^{n_{m_k}} (n_{m_{i+1}})_{i\in F_j}$ and for all j, $(n_{m_{i+1}})_{i\in F_j} \in S_\alpha(N)$. Therefore $(n_{m_{i+1}})_{i\in F} \in S_{\alpha+1}(N)$.

If α is a limit ordinal and $\alpha_n \nearrow \alpha$ are the ordinals used to define S_α and the result holds for all $\beta < \alpha$ (so in particular for each α_n), let $(n_{m_i})_{i\in F} \in S_\alpha$. Thus for some $k \in \mathbf{N}$, $k \leq \min(n_{m_i})_{i\in F} \equiv n_{m_{i_0}} \leq (n_{m_i})_{i\in F} \in S_{\alpha_k}$. Hence $n_k \leq n_{n_{m_{i_0}}} = n_{m_{i_0+1}} \leq (n_{m_{i+1}})_{i\in F} \in S_{\alpha_k}(N)$ therefore $(n_{m_{i+1}})_{i\in F} \in S_\alpha(N)$. ∎

As a corollary we obtain a result of independent interest.

COROLLARY 3.4: *Let $N \in [\mathbf{N}]$. Then there exists $L = (\ell_i) \in [N]$ so that for all $\alpha < \omega_1$,*

$$(\ell_i)_{i\in F} \in S_\alpha \Rightarrow (\ell_i)_{i\in F\setminus \min(F)} \in S_\alpha(N).$$

Proof: Let L be as in Proposition 3.3. Let $F = (f_1 < f_2 < \cdots < f_r)$ with $(\ell_i)_{i\in F} \in S_\alpha$. Thus $(\ell_{f_1+1}, \ell_{f_2+1}, \ldots, \ell_{f_r+1}) \in S_\alpha(N)$. Since $f_1+1 \le f_2$, $f_2+1 \le f_3, \ldots$ and $S_\alpha(N)$ is both spreading and hereditary, we get that $(\ell_i)_{i\in F\setminus \min(F)} \in S_\alpha(N)$. ∎

Proof of Theorem 3.2: Let (x_i) $\Delta_{(e_i)}$- and Δ-stabilize $\bar\gamma$ and γ respectively and let $\alpha < \omega_1$. Since S_α is spreading, $\bar\gamma \le \gamma$. Let $\varepsilon > 0$ and choose $(y_i) \prec (x_i)$ so that for all $(z_i) \prec (y_i)$,

$$|\ \delta_\alpha(z_i) - \gamma_\alpha\ | < \varepsilon.$$

For $i \in \mathbf{N}$ set $n_i = \min(\operatorname{ran}(y_i))$ w.r.t. (e_i) and choose $L = (n_{m_i})$ by Proposition 3.3. For $w \in \operatorname{span}(y_{m_i})$ if $w = \sum_{i=j}^{\ell} a_i y_{m_i}$ where $a_j \ne 0$ we set $\bar w = \sum_{i=j+1}^{\ell} a_i y_{m_i}$.

CLAIM: *If $(w_i)_1^\ell \prec (y_{m_i})$ is α -admissible w.r.t. (e_j) then $(\bar w_i)_1^\ell$ is α -admissible w.r.t. (y_j).*

Indeed let $m_{k_i} = \min(\operatorname{ran}(w_i))$ w.r.t. (y_j). Then $n_{m_{k_i}} = \min(\operatorname{ran}(w_i))$ w.r.t. (e_j), and $(n_{m_{k_i}})_1^\ell \in S_\alpha \Rightarrow (n_{m_{k_i+1}})_1^\ell \in S_\alpha((n_j)) \Rightarrow (m_{k_i+1})_1^\ell \in S_\alpha$. Since $m_{k_i+1} \le \min(\operatorname{ran}(\bar w_i))$ w.r.t. (y_j), and S_α is spreading, the claim follows.

We may assume that $\|y_{m_i}\| = 1$ for all i and that no subsequence of (y_{m_i}) is equivalent to the unit vector basis of c_0 (indeed, if this were false then clearly $\bar\gamma_0 = \gamma_0 = 1$ and $\bar\gamma_\alpha = \gamma_\alpha = 0$ for all $\alpha \ge 1$). Thus by taking long averages of (y_{m_i}) we may choose $(z_i) \prec (y_{m_i})$ with the property that for all $z \in \operatorname{span}(z_i)$

$$\|z - \bar z\| < \varepsilon\|\bar z\|.$$

By the definition of $\bar\delta_\alpha \equiv \delta_\alpha((z_i),(e_i))$ there exists $(w_i)_1^\ell \prec (z_i)$ which is α-admissible w.r.t. (e_j) and satisfies

$$\|\sum_1^\ell w_i\| < (\bar\delta_\alpha + \varepsilon)\sum_1^\ell \|w_i\|.$$

By the above claim $(\bar w_i)_1^\ell$ is α-admissible w.r.t. (y_j). Furthermore

$$\begin{aligned}\|\sum_1^\ell \bar w_i\| &\le \|\sum_1^\ell w_i\| + \sum_1^\ell \|w_i - \bar w_i\| < (\bar\delta_\alpha + \varepsilon)\sum_1^\ell \|w_i\| + \sum_1^\ell \varepsilon\|\bar w_i\| \\ &< [(\bar\delta_\alpha + \varepsilon)(1+\varepsilon) + \varepsilon]\sum_1^\ell \|\bar w_i\|.\end{aligned}$$

It follows that $\gamma_\alpha - \varepsilon < \delta_\alpha(y_i) < (\bar{\gamma}_\alpha + \varepsilon)(1+\varepsilon) + \varepsilon$. Since ε is arbitrary we obtain $\gamma_\alpha \le \bar{\gamma}_\alpha$ and so $\gamma_\alpha = \bar{\gamma}_\alpha$.

To prove that $\Delta(X) = \Delta(X,(e_i))$, let's first show the inclusion $\subseteq$. Let (x_i) Δ-stabilize $\gamma \in \Delta(X)$. We can find $(y_i) \prec (x_i)$ that $\Delta_{(e_i)}$-stabilizes $\bar{\gamma} \in \Delta_{(e_i)}$. But then (y_i) Δ-stabilizes γ, therefore $\gamma = \bar{\gamma}$, thus $\gamma \in \Delta_{(e_i)}$. The inclusion $\supseteq$ is proved similarly. ∎

4. The space $T(\theta_n, S_n)_{\mathbf{N}}$

If $\theta_n \not\to 0$ or if $\theta_n = 1$ for some n then $T(\theta_n, S_n)_{\mathbf{N}}$ is isomorphic to ℓ_1. Thus we shall confine ourselves to the case where $\sup \theta_n < 1$ and $\theta_n \to 0$. Furthermore we assume that $\theta_n \searrow 0$ and $\theta_{m+n} \ge \theta_n\theta_m$ for all $n, m \in \mathbf{N}$. Indeed it is easy to see that $T(\theta_n, S_n)_{\mathbf{N}}$ is naturally isometric to $T(\bar{\theta}_n, S_n)_{\mathbf{N}}$ where

$$\bar{\theta}_n \equiv \sup\left\{\prod_{i=1}^{\ell} \theta_{k_i} \colon \sum_{i=1}^{\ell} k_i \ge n\right\}.$$

Definition 4.1: A sequence (θ_n) of scalars is called **regular** if $(\theta_n) \subset (0,1)$, $\theta_n \searrow 0$ and $\theta_{n+m} \ge \theta_n\theta_m$ for all $n, m \in \mathbf{N}$. If the sequence (θ_n) is regular we define the space $T(\theta_n, S_n)_{\mathbf{N}}$ to be regular.

Throughout this section, the spaces $T(\theta_n, S_n)_{\mathbf{N}}$ will always be assumed to be regular.

It is easy to see (e.g. [OTW]) that if a sequence $(b_n) \subset (0,1]$ satisfies $b_{n+m} \ge b_n b_m$ for all $n, m \in \mathbf{N}$ then $\lim_n b_n^{1/n}$ exists and equals $\sup_n b_n^{1/n}$. Therefore, if the sequence (θ_n) is regular then the limit $\theta \equiv \lim_{n\to\infty} \theta_n^{1/n} = \sup_n \theta_n^{1/n}$ exists. Note also that if $(X, |\!|\!|\cdot|\!|\!|)$ is a Banach space with a basis, then $\delta_{n+m}(X) \ge \delta_n(X)\delta_m(X)$ for all $n, m \in \mathbf{N}$, thus $\lim_n \delta_n(X)^{1/n} = \sup_n \delta_n(X)^{1/n}$ exists. Furthermore, if X does not contain ℓ_1 isomorphically, then $1 > \delta_n(X) \searrow 0$.

For $n \in \mathbf{N}$, define $\phi_n \equiv \theta_n/\theta^n$. We easily see

- If $\theta = 1$ then $\phi_n = \theta_n \searrow 0$.
- $\phi_{n+m} \ge \phi_n\phi_m$ for all $n, m \in \mathbf{N}$.
- $\phi_n^{1/n} \to 1$.
- $\phi_n \le 1, \forall n \in \mathbf{N}$.

From now on, for a regular sequence (θ_n) we will be referring to the limit $\theta = \lim \theta_n^{1/n}$ and the representation $\theta_n = \theta^n \phi_n$ as above.

The main result in this section is the following

THEOREM 4.2: *Let $X = T(\theta_n, S_n)_{\mathbf{N}}$ be regular and let $\theta = \lim_n \theta_n^{1/n}$. Then*

(1) *For all $Y \prec X$, $\ddot{\delta}_1(Y) = \theta$. Moreover for all $\varepsilon > 0$ there exists an equivalent norm $|\cdot|$ on X so that $\delta_1((X, |\cdot|), (e_i)) > \theta - \varepsilon$.*

(2) *For all $Y \prec X$ and for all $n \in \mathbf{N}$, $\ddot{\delta}_n(Y) = \theta^n$ and $\ddot{\delta}_\omega(Y) = 0$.*

(3) *For all $Y \prec X, I_\Delta(Y) = \begin{cases} \omega & \text{if } \theta = 1, \\ 1 & \text{if } \theta < 1. \end{cases}$*

(4) *If $\theta_n/\theta^n \to 0$ then X is arbitrarily distortable.*

To prove the above theorem we need the following two results

PROPOSITION 4.3: *Let $X = T(\theta_n, S_n)_{\mathbf{N}}$ be regular. Then for every $\varepsilon > 0$ there is an equivalent 1-unconditional norm $|\cdot|$ on X such that $\delta_1((X, |\cdot|), (e_i)) \geq \theta - \varepsilon$.*

THEOREM 4.4: *Let $X = T(\theta_n, S_n)_{\mathbf{N}}$ be regular. Then for all $Y \prec X$ and $j \in \mathbf{N}$ we have*

$$\delta_j(Y) \leq \theta^{j-1} \sup_{p \geq j} \phi_p \vee \frac{\theta_j}{\theta_1}.$$

Proof of Theorem 4.2: (1) To prove $Y \prec X$ implies $\ddot{\delta}_1(Y) \leq \theta$ we note that if $|\!|\!|\cdot|\!|\!|$ is an equivalent norm on Y then there exists $C \geq 1$ such that $C^{-1}\delta_n(Y) \leq \delta_n(Y, |\!|\!|\cdot|\!|\!|) \leq C\delta_n(Y)$ for all $n \in \mathbf{N}$. Let $\delta_n \equiv \delta_n(Y, |\!|\!|\cdot|\!|\!|)$. Then since for all n and m, $\delta_{n+m} \geq \delta_n \delta_m$, we have $\lim_n \delta_n^{1/n} = \sup_n \delta_n^{1/n}$ exists. Hence $\delta_1 \leq \lim \delta_n^{1/n} = \lim \delta_n(Y)^{1/n}$, the latter limit existing for the same reason. Now

$$\lim \delta_n(Y)^{1/n} \leq \lim_{n \to \infty} \left(\theta^{n-1} \sup_{p \geq n} \phi_p \vee \frac{\theta_n}{\theta_1}\right)^{1/n} = \theta$$

by Theorem 4.4. Thus $\ddot{\delta}_1(Y) \leq \theta$ as required. The "moreover" part is Proposition 4.3 and this completes the proof of $\ddot{\delta}_1(Y) = \theta$.

(2) Since $\ddot{\delta}_1(Y) = \theta$ we obtain $\ddot{\delta}_n(Y) = \theta^n$ from Theorem 2.5. By Theorem 4.4 we have that for all $\gamma \in \Delta(Y)$ and for all $j \in \mathbf{N}$,

$$\gamma_j \leq \theta^{j-1} \sup_{p \geq j} \phi_p \vee \frac{\theta_j}{\theta_1}.$$

Therefore, again by Theorem 2.5, for all $\gamma \in \Delta(Y)$, $\gamma_\omega = \lim_{n \in \mathbf{N}} \gamma_n = 0$. Hence, for every equivalent norm $|\cdot|$ on Y, for every $\gamma \in \Delta(Y, |\cdot|)$, $\gamma_\omega = 0$. Since $\ddot{\delta}_\omega(Y) = \sup\{\gamma_\omega : \gamma \in \ddot{\Delta}(Y)\}$ we have $\ddot{\delta}_\omega(Y) = 0$.

(3) Follows immediately from (2).

(4) Let $\lambda > 1$, and choose $n \in \mathbf{N}$ so that $\sup_{p \geq n} \phi_p < \theta_1/2\lambda$. By (1) we can define an equivalent norm $|\!|\!|\cdot|\!|\!|$ on X such that

$$\delta_n((X, |\!|\!|\cdot|\!|\!|), (e_i)) \geq \delta_1((X, |\!|\!|\cdot|\!|\!|), (e_i))^n \geq \theta^n/2.$$

Let $Y \prec X$; then by equation (2) of section 2, there exists $C > 0$ such that

$$C|||y||| \leq \|y\| \leq D(Y, |||\cdot|||)C|||y|||, \qquad \text{for all } y \in Y.$$

Therefore by Observation 2.2,

$$D(Y, |||\cdot|||) \geq \frac{\delta_n(Y, |||\cdot|||)}{\delta_n(Y, \|\cdot\|)}.$$

Since $\delta_n(Y, |||\cdot|||) \geq \delta_n((X, |||\cdot|||), (e_i)) \geq \theta^n/2$, and

$$\delta_n(Y, \|\cdot\|) \leq \theta^{n-1} \sup_{p\geq n} \phi_p \vee \frac{\theta_n}{\theta_1} \leq \frac{1}{\theta_1}\theta^n \sup_{p\geq n} \phi_p$$

(by Theorem 4.4), we obtain

$$D(Y, |||\cdot|||) \geq \frac{\theta_1}{2\sup_{p\geq n} \phi_p} > \lambda. \qquad \blacksquare$$

The proof of Proposition 4.3 comes from an argument in [OTW]. We recall this argument here.

Sketch of the proof of Proposition 4.3: Fix $n \in \mathbf{N}$ such that $\theta_n^{1/n} > \theta - \varepsilon$ and set $a \equiv \theta_n^{1/n}$. For $j \in \mathbf{N}$ and $x \in X$ define

$$| x |_j = \sup\{a^j \sum_1^{\ell} \|E_i x\| \colon (E_i x)_1^{\ell} \text{ is } j\text{-admissible w.r.t. } (e_i)\} \quad \text{and}$$

$$| x | = \frac{1}{n} \sum_{j=0}^{n-1} | x |_j \quad (\text{where } | \cdot |_0 = \|\cdot\|).$$

We claim that $\delta_1((X, |\cdot|), (e_i)) \geq a$. To see this let $e_k \leq x_1 < x_2 < \cdots < x_k$ in X and $x = \sum_{i=1}^k x_i$. For $j = 1, \ldots, n-1$ we have $| x |_j \geq a \sum_{i=1}^k | x_i |_{j-1}$ (by the definitions of $|\cdot|_j$ and $|\cdot|_{j-1}$) and also $| x |_0 \geq a \sum_{i=1}^k | x_i |_{n-1}$ (since $a^n = \theta_n$). Therefore we get $| x | \geq a \sum_{i=1}^k | x_i |$. $\blacksquare$

To prove Theorem 4.4 we need some norm estimates in $T(\theta_n, S_n)_{\mathbf{N}}$ for certain iterated rapidly increasing averages. Before defining what we mean by this we fix some terminology.

Let E and F be intervals in $\mathbf{N}$. We say that E **does not split** F if either $E \cap F = \emptyset$ or $F \subseteq E$. For $x \in c_{00}$ E does not split x if E does not split $\mathrm{ran}(x)$. Let (x_i) be a block basis of (e_i) in c_{00}, and let $E_1 < E_2 < \cdots < E_N$ be intervals in $\mathbf{N}$. We say that we **minimally shrink the intervals** $(E_\ell)_{\ell=1}^N$

to obtain intervals $(F_\ell)_{\ell=1}^n$ **which don't split the** x_i**'s**, if for $\ell = 1, \ldots, N$ we let $G_\ell = E_\ell \setminus \cup \{\mathrm{ran}(x_i)\colon E_\ell \text{ splits } x_i\}$ and let $F_1 < F_2 < \cdots < F_n$ be the enumeration of the non-empty G_ℓ's.

By a **tree** we shall mean a non-empty partially ordered set $(\mathcal{T}, \ll)$ for which the set $\{y \in \mathcal{T}\colon y \ll x\}$ is linearly ordered and finite for each $x \in \mathcal{T}$. If $\mathcal{T}' \subseteq \mathcal{T}$ then we say that $(\mathcal{T}', \ll)$ is a **subtree** of $(\mathcal{T}, \ll)$. The tree $\mathcal{T}$ is called **finite** if the set $\mathcal{T}$ is finite. The **initial** nodes of $\mathcal{T}$ are the minimal elements of $\mathcal{T}$ and the **terminal** nodes are the maximal elements. A **branch** in $\mathcal{T}$ is a maximal linearly ordered set in $\mathcal{T}$. The **immediate successors** of $x \in \mathcal{T}$ are all the nodes $y \in \mathcal{T}$ such that $x \ll y$ but there is no $z \in \mathcal{T}$ with $x \ll z \ll y$. If X is a linear space, then a **tree in** X is a tree whose nodes are vectors in X. If X is a Banach space with a basis (e_i) and $(x_i) \prec (e_i)$ then an **admissible averaging tree of** (x_i) is a finite tree $\mathcal{T}$ in X with the following properties:

- $\mathcal{T} = (x_i^j)_{j=0,i=1}^{M,N^j}$ where $M \in \mathbf{N}$ and $1 = N^M \leq \cdots \leq N^1 \leq N^0$.
- $x_1^j < \cdots < x_{N^j}^j$ w.r.t. (e_s) $(j = 0, 1, \ldots, M-1)$ and $(x_i^0)_{i=1}^{N^0}$ is a subsequence of (x_s).

Also for $j = 1, \ldots, M$ and $i = 1, \ldots, N^j$ we have the following:

- There exists a non-empty interval $I_i^j \subseteq \{1, \ldots, N^{j-1}\}$ such that $\{x_s^{j-1}\colon s \in I_i^j\}$ are the immediate successors of x_i^j.
- $x_i^j = \dfrac{1}{|I_i^j|} \sum_{s \in I_i^j} x_s^{j-1}$.
- $(\min(\mathrm{ran}(x_s^{j-1})))_{s \in I_i^j} \in S_1$ where $\mathrm{ran}(x_s^{j-1})$ is taken w.r.t. (x_s).

Note that the last three properties together require that x_i^j be a 1-admissible average of all of its immediate successors w.r.t. (x_s). Let $\mathcal{T} = (x_i^j)_{j=0,i=1}^{M,N^j}$ be an admissible averaging tree as in the above definition, and let $b = \{y_M \ll \cdots \ll y_0\}$ be a branch in $\mathcal{T}$. For $i = 0, 1, \ldots, M$ we say that the **level** of y_i is i. Note that this is well defined, since the definition of admissible averaging trees forces every branch to have the same number of elements. Indeed for each i and j, the level of x_i^j in $\mathcal{T}$ is j. Let $\mathcal{T}$ be a tree, $x \in \mathcal{T}$ of level ℓ and $k \in \mathbf{N}$. By $\mathcal{T}(x,k)$ (resp. $\mathcal{T}^*(x,k)$) we shall denote the subtree of $\mathcal{T}' = \{x\} \cup \{y \in \mathcal{T}\colon y \gg x\}$ (resp. $\mathcal{T}' = \{y \in \mathcal{T}\colon y \gg x\}$) which contains all the nodes of $\mathcal{T}'$ that have level $\ell, \ell-1, \ldots,$ or $\ell-k+1$ in $\mathcal{T}$. Let $\mathcal{T}$ be an admissible averaging tree in a Banach space X with a basis (e_i). Let $x \in \mathcal{T}$ have immediate successors $x_1 < \cdots < x_n$ (a finite block basis of (e_i)), let $k \in \mathbf{N}$, and let $F \subseteq \mathbf{N}$ be an interval which does not split any of $x_1,\dots,x_n$. Then by $\mathcal{T}_F(x,k)$ we shall denote the subtree of $\mathcal{T}(x,k)$ given by $\mathcal{T}_F(x,k)=\{x\}\cup\{y\in\mathcal{T}^*(x,k)\colon \operatorname{ran}(y)$ (w.r.t. (e_i)) $\subseteq F\}$.

Definition 4.5: Let (x_i) be a block sequence of (e_i) in c_{00}, $M,N\in\mathbf{N}$, and let $(\varepsilon_i^j)_{j,i\in\mathbf{N}}\cup\{\theta\}\subset(0,1)$. We say that x is an $(M,(\varepsilon_i^j),\theta,N)$ average of (x_i) w.r.t. (e_i) if there exists an admissible averaging tree $\mathcal{T}=(x_i^j)_{j=0,i=1}^{M,N^j}$ of (x_i) whose initial node is x $(=x_1^M)$ and

> for $j=1,\dots,M$ and $1\le i\le N^j$ if $N_i^j=\max(\operatorname{ran}(x_i^j))$ w.r.t. (e_s) $(N_0^j=N)$, then x_i^j is an average of its immediate successors of length $k_i^j>6N_{i-1}^j/\theta\varepsilon_i^j$.

$\mathcal{T}$ then will be called an $(M,(\varepsilon_i^j),\theta,N)$ admissible averaging tree of (x_i) w.r.t. (e_i). For $i=1,\dots,N^0$ set $N_i^0=\max(\operatorname{ran}(x_i^0))$ w.r.t. (e_s), and $N_0^0=N$. Then $(N_i^j)_{j=0,i=0}^{M,N^j}$ are called the maximum coordinates of $\mathcal{T}$ w.r.t. (e_i).

Remark 4.6: Let X be a Banach space with basis (e_i) and let (x_i) be a block sequence of (e_i) with $\|x_i\|\le 1$ for all $i\in\mathbf{N}$. Let $(\varepsilon_i^j)_{j,i\in\mathbf{N}}\cup\{\theta\}\subset(0,1)$. Let $M,N\in\mathbf{N}$ and let x be an $(M,(\varepsilon_i^j),\theta,N)$ average of (x_i) w.r.t. (e_i) given by $\mathcal{T}=(x_i^j)_{j=0,i=1}^{M,N^j}$. Then we can write $x=\sum_{i\in F}a_ix_i$ for some finite set $F\subset\mathbf{N}$ such that

(1) $\sum_{i\in F}a_i=1$ & $a_i>0$ for all $i\in F$.
(2) x is M-admissible w.r.t. (x_i) (i.e. $F\in S_M$).
(3) Let $(N_i^j)_{j=0,i=0}^{M,N^j}$ be the maximum coordinates of $\mathcal{T}$ w.r.t. (e_s). For $j=1,\dots,M$ and $1\le i\le N^j$, let $E_i^j(1)<E_i^j(2)<\cdots<E_i^j(N_{i-1}^j)$ be a finite sequence of intervals in $\mathbf{N}$ with $\bigcup_{\ell=1}^{N_{i-1}^j}E_i^j(\ell)\subseteq\operatorname{ran}(x_i^j)$ and assume that we minimally shrink the $E_i^j(\ell)$'s to obtain intervals $(F_i^j(\ell))_{\ell=1}^{N_{i-1}^j}$ (some of which may be empty) which don't split the x_i^{j-1}'s. Then
$$\sum_{j=1}^M\sum_{i=1}^{N^j}\sum_{\ell=1}^{N_{i-1}^j}\|(E_i^j(\ell)\setminus F_i^j(\ell))x_i^j\|<\sum_{j,i}\varepsilon_i^j.$$

Indeed (1) and (2) are obvious. To see (3) note that for every $j=1,\dots,M$, $1\le i\le N^j$ and $\ell=1,\dots,N_{i-1}^j$, the set $E_i^j(\ell)$ splits at most two x_s^{j-1}'s each of them having norm at most 1. Thus $\|(E_i^j(\ell)\setminus F_i^j(\ell))x_i^j\|\le 2/k_i^j$ and so $\sum_{\ell=1}^{N_{i-1}^j}\|(E_i^j(\ell)\setminus F_i^j(\ell))x_i^j\|<2N_{i-1}^j/k_i^j<\varepsilon_i^j$, which proves (3).

The concept of $(M,(\varepsilon_i^j),N)$ averages is implicit in [AD] (see also [OTW]).

PROPOSITION 4.7: *Let (x_i) be a block sequence in c_{00}, $M,N\in\mathbf{N}$ and $(\varepsilon_i^j)_{j,i\in\mathbf{N}}\cup\{\theta\}\subset(0,1)$. Then there exists x which is an $(M,(\varepsilon_i^j),\theta,N)$ average of (x_i) w.r.t. (e_i).*

Proof: Note that by replacing each $(\varepsilon_i^j)_i$ by a smaller sequence if necessary we may assume that $(\varepsilon_i^j)_i$ is decreasing. For $M = 1$ we choose x_1^1 to be an average of $k_1^1 > 6N/(\theta\varepsilon_1^1)$ many x_s's chosen from $\{x_s : s \geq k_1^1\}$. Next, consider the case $M = 2$. At first we continue the argument that we gave for $M = 1$ to construct $\bar{x}_1^1 < \bar{x}_2^1 < \cdots$ as follows: For $\bar{k}_1^1 > 6N/(\theta\varepsilon_1^1)$ let $\bar{x}_1^1$ be an average of $\bar{k}_1^1$ many x_s's chosen from $\{x_s : s \geq \bar{k}_1^1\}$. If $\bar{x}_i^1$ has been constructed for some $i \in \mathbf{N}$, and $\bar{k}_{i+1}^1 > 6\bar{N}_i^1/(\theta\varepsilon_{i+1}^1)$, then $\bar{x}_{i+1}^1$ is taken to be an average of $\bar{k}_{i+1}^1$ many x_s's chosen from $\{x_s : s \geq \bar{k}_{i+1}^1\}$ where $\bar{N}_i^1 = \max(\operatorname{ran}(\bar{x}_i^1))$ w.r.t. (e_s). Note that $\bar{x}_i^1 < \bar{x}_{i+1}^1$ since $\varepsilon_{i+1}^1 < 1$. Also note that for every $i \in \mathbf{N}$, $\bar{x}_i^1$ is a 1-admissible w.r.t. (x_s). Then for $k_1^2 > 6N/(\theta\varepsilon_1^2)$ take x_1^2 to be an average of k_1^2 many $\bar{x}_s^1$'s chosen from $\{\bar{x}_s^1 : \bar{x}_s^1 \geq x_{k_1^2}\}$. Then the $(2, (\varepsilon_i^j), \theta, N)$ admissible averaging tree $\mathcal{T}$ of (x_i) that corresponds to x_1^2 is determined as follows: $x_1^2 \in \mathcal{T}$. If $x_1^2 = \frac{1}{|F|}\sum_{i\in F}\bar{x}_i^1$ for some finite set $F \subset \mathbf{N}$ then $\bar{x}_i^1 \in \mathcal{T}$ for $i \in F$. For each $i \in F$ if $\bar{x}_i^1 = \frac{1}{|F_i|}\sum_{s\in F_i} x_s$ for some finite set $F_i \subset \mathbf{N}$ then $x_s \in \mathcal{T}$ for $s \in F_i$. Enumerate the x_s's in $\mathcal{T}$ as $x_1^0 < x_2^0 < \cdots < x_{N^0}^0$ and the $\bar{x}_s^1$'s in $\mathcal{T}$ as $x_1^1 < x_2^1 < \cdots < x_{N^1}^1$. Since x_1^2 is a 1-admissible average of (x_i^1) w.r.t. (x_i) and for each $i = 1, \ldots, N^1$, x_i^1 is 1-admissible w.r.t. (x_s), we have that x_1^2 is 2-admissible w.r.t. (x_i). We let the k_i^1's and N_i^1's be defined by Definition 4.5. Each k_i^1 will be $\bar{k}_{i'}^1$ for some $i' \geq i$ and $N_0^1 = N$ while $N_i^1 = N_{i'}^1$. Since (ε_i^1) is decreasing the condition $k_i^1 > 6N_{i-1}^1/(\theta\varepsilon_i^1)$ remains valid. The case $M > 2$ is proved by iterating this procedure. ∎

Next we prove some norm estimates for $(M, (\varepsilon_i^j)\theta_1, N)$ averages in $T(\theta_n, S_n)_{\mathbf{N}}$. We will always denote the norm of $T(\theta_n, S_n)_{\mathbf{N}}$ by $\|\cdot\|$. We need for $p \in \mathbf{N}\cup\{0\}$ and $N \in \mathbf{N}$ to define the equivalent norms $\|\cdot\|_p$ and $\|\cdot\|_{S_N,p}$ and the continuous seminorms $\|\cdot\|_{N,p}$ as follows ($\|\cdot\|_0 = \|\cdot\|$ and $\theta_0 = 1$):

$$\|x\|_p = \theta_p \sup\{\sum \|E_i x\| \colon (E_i) \text{ is a } p\text{-admissible sequence of intervals }\},$$

$$\|x\|_{N,p} = \sup\{\sum_1^N \|E_i x\|_p \colon N \leq E_1 < E_2 < \cdots < E_N \text{ are intervals }\} \quad \text{and}$$

$$\|x\|_{S_N,p} = \sup\{\sum \|E_i x\|_p \colon (E_i) \text{ is } N\text{-admissible sequence of intervals}\}.$$

Of course for $x \in c_{00}$ each "sup" above is a "max" and there exists $p \in \mathbf{N}$ so that $\|x\| = \|x\|_p$ if $\|x\| \neq \|x\|_\infty$.

Remark 4.8: Let $\theta_0 = 1$. For all $x \in c_{00}$ and for all $p \in \mathbf{N}$ we have

$$\|x\|_p \leq \frac{\theta_p}{\theta_{p-1}}\|x\|_{S_1,p-1}.$$

Moreover, if $p = 1$ we have equality.

Indeed there exists $(E_i)_{i\in I}$ a p-admissible family of intervals such that

$$\|x\|_p = \theta_p \sum_{i\in I} \|E_i x\|.$$

We can write $I = \bigcup_1^\ell I_j$ where $(E_i)_{i\in I_j}$ is $p-1$-admissible and if F_j is the smallest interval including $\bigcup_{i\in I_j} E_i$ then $(F_j)_1^\ell$ is 1-admissible. Thus

$$\|x\|_p = \frac{\theta_p}{\theta_{p-1}} \sum_{j=1}^{\ell} \theta_{p-1} \sum_{i\in I_j} \|E_i x\| \le \frac{\theta_p}{\theta_{p-1}} \sum_{j=1}^{\ell} \|F_j x\|_{p-1} \le \frac{\theta_p}{\theta_{p-1}} \|x\|_{S_1,p-1}. \qquad \blacksquare$$

Notation: If $A \subset [0,\infty)$ is a finite non-empty set, we set $A^* = A\backslash\{\max(A)\}$.

Observation 4.9: Let $N \in \mathbf{N}$ and $D,\varepsilon > 0$. Note that if $k \ge ND/\varepsilon$ and $A_\ell \subset [0,D]$ for $\ell = 1,\dots,N$ are finite sets with $|A_1| + \cdots + |A_N| \le k$ then $\frac{1}{k}\sum_{\ell=1}^N \sum\{a\colon a \in A_\ell\} \le \max(\bigcup_{\ell=1}^N A_\ell^*) + \varepsilon$.

We will apply this for $D = 1/\theta_1$ in the proof of *(2)* of Lemma 4.10 below.

LEMMA 4.10: *Let $x_1,\dots,x_k$ be non-zero vectors with $k \le x_1 < x_2 < \cdots < x_k$ and $\|x_i\| \le 1$ for all i, let $x = \frac{1}{k}(x_1 + \cdots + x_k)$ and $\varepsilon \in (0,1)$. Let $F \subseteq \mathrm{ran}(x)$ be an interval in $\mathbf{N}$ which does not split the x_i's. Set $\theta_0 = 1$, $N_i = \max(\mathrm{ran}(x_i))$ w.r.t. (e_j), $N_0 = 1$ and let $N \in \mathbf{N}$. If $k > 6N/\theta_1\varepsilon$ then*

(1) *For every $p \in \mathbf{N}$,*

$$\|Fx\|_{N,p} \le \frac{\theta_p}{\theta_{p-1}} \max\{\|x_i\|_{N_{i-1},p-1}\colon \mathrm{ran}(x_i) \subseteq F\} + \varepsilon.$$

(2) *There exists $n \in \mathbf{N}$, intervals $F_1 < F_2 < \cdots < F_n$ which don't split any x_i, $\bigcup_{\ell=1}^n F_\ell \subseteq \mathrm{ran}(x)$, and $(p_\ell)_{\ell=1}^n \subset \mathbf{N}$ so that*

$$\|x\|_{N,0} \le \max\left(\bigcup_{\ell=1}^{n} \left\{\frac{\theta_{p_\ell}}{\theta_{p_\ell-1}} \|x_i\|_{N_{i-1},p_\ell-1}\colon \mathrm{ran}(x_i) \subset F_\ell\right\}^*\right) + \varepsilon.$$

Proof: (1) For $p \in \mathbf{N}$ there exist intervals $N \le E_1 < \cdots < E_N$ such that $\bigcup_{\ell=1}^N E_\ell \subseteq F$ and

$$\|Fx\|_{N,p} = \sum_{\ell=1}^{N} \|E_\ell x\|_p \le \frac{\theta_p}{\theta_{p-1}} \sum_{\ell=1}^{N} \|E_\ell x\|_{S_1,p-1} \quad \text{(by Remark 4.8)}.$$

We minimally shrink the intervals $(E_i)_1^N$ to get $n \le N$ and intervals $N \le F_1 < F_2 < \cdots < F_n$ which don't split the x_i's. Since each E_ℓ splits at most two x_i's,

$$\|\cdot\|_{S_1,p-1} \le \frac{1}{\theta_1}\|\cdot\| \quad \text{and} \quad \frac{\theta_p}{\theta_{p-1}} \le 1,$$

$$\frac{\theta_p}{\theta_{p-1}}\sum_{\ell=1}^{N}\|E_\ell x\|_{S_1,p-1}\le\frac{\theta_p}{\theta_{p-1}}\sum_{\ell=1}^{n}\|F_\ell x\|_{S_1,p-1}+\frac{2N}{k\theta_1}.$$

Fix an $\ell\in\{1,\dots,n\}$. There exists a 1-admissible family of intervals $(F_{\ell,m})_m$ with $F_{\ell,m}\subseteq F_\ell$ for all m and $\|F_\ell x\|_{S_1,p-1}=\sum_m\|F_{\ell,m}x\|_{p-1}$.

Let s be minimal with $\mathrm{ran}(x_s)\cap F_{\ell,1}\ne\emptyset$ (we may assume that such an s exists) and t be maximal with $\mathrm{ran}(x_t)\cap F_\ell\ne\emptyset$. Then

$$\begin{aligned}\sum_m\|F_{\ell,m}x\|_{p-1}&\le\frac{1}{k}(\sum_m\|F_{\ell,m}x_s\|+\|x_{s+1}\|_{N_s,p-1}+\dots+\|x_t\|_{N_s,p-1})\\&\le\frac{1}{k}(\frac{1}{\theta_1}+\sum_{i=s+1}^{t}\|x_i\|_{N_{i-1},p-1}).\end{aligned}$$

Set $[r,R]\equiv\{i\colon \mathrm{ran}(x_i)\subseteq F\}$. Hence

$$\frac{\theta_p}{\theta_{p-1}}\sum_{\ell=1}^{n}\|F_\ell x\|_{S_1,p-1}\le$$

$$\frac{\theta_p}{\theta_{p-1}}\frac{1}{k}\Big(\frac{n}{\theta_1}+\|x_{r+1}\|_{N_r,p-1}+\|x_{r+2}\|_{N_{r+1},p-1}+\dots+\|x_R\|_{N_{R-1},p-1}\Big).$$

Therefore we have proved that

$$\begin{aligned}\|Fx\|_{N,p}\le&\frac{1}{k}\frac{\theta_p}{\theta_{p-1}}\big(\|x_{r+1}\|_{N_r,p-1}+\|x_{r+2}\|_{N_{r+1},p-1}+\dots+\|x_R\|_{N_{R-1},p-1}\big)\\&+\frac{3N}{k\theta_1},\end{aligned}$$

thus

$$\|Fx\|_{N,p}\le\frac{1}{k}\frac{\theta_p}{\theta_{p-1}}\sum\{\|x_i\|_{N_{i-1},p-1}\colon \mathrm{ran}(x_i)\subseteq F\}+\frac{3N}{k\theta_1},\tag{3}$$

which yields (1) of Lemma 4.10.

(2) Choose intervals $N\le E_1<E_2<\dots<E_N$ such that $\|x\|_{N,0}=\sum_{l=1}^{N}\|E_\ell x\|$. As before, we minimally shrink the intervals (E_i) to obtain $n\le N$ and non-empty intervals $F_1<F_2<\dots<F_n$ which don't split the x_i's and satisfy

$$\sum_{\ell=1}^{N}\|E_\ell x\|\le\sum_{\ell=1}^{n}\|F_\ell x\|+\frac{2N}{k}.$$

Fix $\ell\in\{1,\dots,n\}$. If $\|F_\ell x\|\ne\|F_\ell x\|_\infty$ then there exists $p_\ell\in\mathbf{N}$ such that $\|F_\ell x\|=\|F_\ell x\|_{p_\ell}$. By equation (3) for $N=1$ we get

$$\|F_\ell x\|_{p_\ell}\le\frac{1}{k}\frac{\theta_{p_\ell}}{\theta_{p_\ell-1}}\Big(\sum\{\|x_i\|_{N_{i-1},p_\ell-1}\colon \mathrm{ran}(x_i)\subseteq F_\ell\}\Big)+\frac{3}{k\theta_1}.\tag{4}$$

If $\|F_\ell x\| = \|F_\ell x\|_\infty$ then $\|F_\ell x\| \le 1/k$ and so (4) still is valid. Thus

$$\begin{aligned}\|x\|_{N,0} &\le \frac{1}{k}\sum_{\ell=1}^{n}\frac{\theta_{p_\ell}}{\theta_{p_\ell-1}}\sum\{\|x_i\|_{N_{i-1},p_\ell-1}\colon \operatorname{ran}(x_i)\subseteq F_\ell\} + \frac{5N}{k\theta_1}\\ &< \max\left(\bigcup_{\ell=1}^{n}\{\frac{\theta_{p_\ell}}{\theta_{p_\ell-1}}\|x_i\|_{N_{i-1},p_\ell-1}\colon \operatorname{ran}(x_i)\subset F_\ell\}^*\right)+\varepsilon\end{aligned}$$

by Observation 4.9, since $\|\cdot\|_{N_{i-1},p_\ell-1} \le 1/\theta_1\|\cdot\|$, and $k > 6N/\varepsilon\theta_1 = N/(\varepsilon/6)\theta_1$. ∎

LEMMA 4.11: *Let (x_i) be a normalized block sequence in $X = T(\theta_i, S_i)_{\mathbf{N}}$, $M, N \in \mathbf{N}$ and $(\varepsilon_i^j)_{j,i\in\mathbf{N}} \subset (0,1)$. Let x be an $(M, (\varepsilon_i^j), \theta_1, N)$ average of (x_i) w.r.t. (e_i), let $\mathcal{T} = (x_i^j)_{j=0,i=1}^{M,N^j}$ be the corresponding admissible averaging tree of (x_i) with $x = x_1^M$, and let $(N_i^j)_{j=0,i=0}^{M,N^j}$ be the maximum coordinates of $\mathcal{T}$ w.r.t. (e_i). Then for $j = 1,\dots,M$ and $i = 1,\dots,N^j$ we have the following properties:*

(1) *For every $p \in \mathbf{N}$ and every $F \subseteq \operatorname{ran}(x_i^j)$ which does not split any x_s^{j-1} we have*

$$\|Fx_i^j\|_{N_{i-1}^j,p} \le \frac{\theta_p}{\theta_{p-1}}\max\{\|x_s^{j-1}\|_{N_{s-1}^{j-1},p-1}\colon \operatorname{ran}(x_s^{j-1})\subseteq F\} + \varepsilon_i^j/N_{i-1}^j.$$

(2) *There exists $n \in \mathbf{N}$ and intervals $F_1 < F_2 < \cdots < F_n$ which don't split any x_s^{j-1}, ($\bigcup_{\ell=1}^n F_\ell \subseteq \operatorname{ran}(x_i^j)$) and $(p_\ell)_{\ell=1}^n \subseteq \mathbf{N}$ such that*

$$\|x_i^j\|_{N_{i-1}^j,0} \le \max\left(\bigcup_{\ell=1}^{n}\left\{\frac{\theta_{p_\ell}}{\theta_{p_\ell-1}}\|x_s^{j-1}\|_{N_{s-1}^{j-1},p_\ell-1}\colon \operatorname{ran}(x_s^{j-1})\subseteq F_\ell\right\}^*\right)+\varepsilon_i^j.$$

(3) *If $0 \le p' < p$, $p - p' \le j \le M$, $1 \le i \le N^j$ and $F \subseteq \operatorname{ran}(x_i^j)$ is an interval which does not split any x_s^{j-1} then*

$$\begin{aligned}\|Fx_i^j\|_{N_{i-1}^j,p} \le& \frac{\theta_p}{\theta_{p'}}\max\{\|x_s^{j-(p-p')}\|_{N_{s-1}^{j-(p-p')},p'}\colon \operatorname{ran}(x_s^{j-(p-p')})\subseteq F\}\\ &+\sum\left\{\frac{\varepsilon_s^k}{N_{s-1}^k}\colon x_s^k\in\mathcal{T}_F(x_i^j,p-p')\right\}.\end{aligned}$$

(4) *If $1 \le p \le j \le M$, $1 \le i \le N^j$ and $F \subseteq \mathbf{N}$ is an interval which does not split any x_s^{j-1} then*

$$\begin{aligned}\|Fx_i^j\|_{N_{i-1}^j,p} \le& \theta_p\max\{\|x_s^{j-p}\|_{N_{s-1}^{j-p},0}\colon \operatorname{ran}(x_s^{j-p})\subseteq F\}\\ &+\sum\left\{\frac{\varepsilon_s^k}{N_{s-1}^k}\colon x_s^k\in\mathcal{T}_F(x_i^j,p)\right\}.\end{aligned}$$

(5) *There exists $m \in \mathbf{N}$ and intervals $F_1 < F_2 < \cdots < F_m$ ($\bigcup_\ell F_\ell \subseteq \operatorname{ran}(x_1^M)$) which don't split the x_s^0's and $(p_\ell)_{\ell=1}^m \subset \mathbf{N}$ with $p_\ell \geq M$ for all ℓ, such that*

$$\|x_1^M\| \leq \max\left(\bigcup_{\ell=1}^m \left\{\frac{\theta_{p_\ell}}{\theta_{p_\ell - M}}\|x_s^0\|_{N_{s-1}^0, p_\ell - M}\colon \operatorname{ran}(x_s^0) \subseteq F_\ell\right\}^*\right) + \varepsilon.$$

(6) *For $J = 1, \dots, M$, $1 \leq i \leq N^J$,*

$$\|x_i^J\|_{N_{i-1}^J, 0} \leq \theta^{J-1} + \varepsilon.$$

(7) *If $1 \leq p \leq M$ then $\|x\|_{N,p} \leq \phi_p \theta^{M-1} + \varepsilon$.*

Proof: (1), (2) Combine Lemma 4.10 with Definition 4.5.

(3) By (1) of Lemma 4.11 we have

$$\begin{aligned}
\|Fx_i^j\|_{N_{i-1}^j, p} \leq& \frac{\theta_p}{\theta_{p-1}} \max\{\|x_s^{j-1}\|_{N_{s-1}^{j-1}, p-1}\colon \operatorname{ran}(x_s^{j-1}) \subseteq F\} + \frac{\varepsilon_i^j}{N_{i-1}^j} \\
\leq& \frac{\theta_p}{\theta_{p-1}} \frac{\theta_{p-1}}{\theta_{p-2}} \max\{\|x_s^{j-2}\|_{N_{s-1}^{j-2}, p-2}\colon \operatorname{ran}(x_s^{j-2}) \subseteq F\} \\
&+ \sum\left\{\frac{\varepsilon_s^k}{N_{s-1}^k}\colon x_s^k \in \mathcal{T}_F(x_i^j, 2)\right\} \\
\leq& \cdots \\
\leq& \frac{\theta_p}{\theta_{p-1}} \frac{\theta_{p-1}}{\theta_{p-2}} \cdots \frac{\theta_{p'+1}}{\theta_{p'}} \\
&\times \max\{\|x_s^{j-(p-p')}\|_{N_{s-1}^{j-(p-p')}, p'}\colon \operatorname{ran}(x_s^{j-(p-p')}) \subseteq F\} \\
&+ \sum\left\{\frac{\varepsilon_s^k}{N_{s-1}^k}\colon x_s^k \in \mathcal{T}_F(x_i^j, p-p')\right\}.
\end{aligned}$$

(4) Follows immediately from (3), letting $p' = 0$.

(5) We prove by induction on J that

(5′) for $J = 1, \dots, M$ and $1 \leq i \leq N^J$ there exists $m \in \mathbf{N}$, intervals $F_1 < F_2 < \cdots < F_m$ ($\bigcup_\ell F_\ell \subseteq \operatorname{ran}(x_i^J)$) that don't split the x_s^0's, and $(p_\ell)_{\ell=1}^m \subset \mathbf{N}$ with $p_\ell \geq J$ for all ℓ, such that

$$\begin{aligned}
\|x_i^J\|_{N_{i-1}^J, 0} \leq& \max\left(\bigcup_{\ell=1}^m \left\{\frac{\theta_{p_\ell}}{\theta_{p_\ell - J}}\|x_s^0\|_{N_{s-1}^0, p_\ell - J}\colon \operatorname{ran}(x_s^0) \subseteq F_\ell\right\}^*\right) \\
&+ \sum\{\varepsilon_s^k\colon x_s^k \in \mathcal{T}(x_i^J, J)\}.
\end{aligned}$$

((5) then follows by taking $(J,i)=(M,1)$ and noting that $\|x_1^M\| \le \|x_1^M\|_{N,0} = \|x\|_{N_0^M,0}$.) Indeed, for $J=1$ this follows from the statement of (2) for $j=1$. Assume that the statement is proved for all positive integers $\le J$ where $J \le M-1$. By (2) there exist intervals $F_1' < \cdots < F_n'$ ($\bigcup_\ell F_\ell' \subseteq \operatorname{ran}(x_i^{J+1})$) which don't split the x_s^J's, and $(p_\ell')_{\ell=1}^n$ such that

$$\|x_i^{J+1}\|_{N_{i-1}^{J+1},0} \le \max\left(\bigcup_{\ell=1}^n \left\{\frac{\theta_{p_\ell'}}{\theta_{p_\ell'-1}}\|x_s^J\|_{N_{s-1}^J,p_\ell'-1}\colon \operatorname{ran}(x_s^J)\subseteq F_\ell'\right\}^*\right) + \varepsilon_i^{J+1}.$$

If $p_\ell'-1=0$ for some ℓ and $\operatorname{ran}(x_s^J)\subseteq F_\ell'$ then by the induction hypothesis there exists $M(s)\in\mathbf{N}$, intervals $F_1(s)<F_2(s)<\cdots<F_{M(s)}(s)$ ($\bigcup_\mu F_\mu(s)\subseteq \operatorname{ran}(x_s^J)$) which don't split the x_t^0's and $(p_\mu(s))_{\mu=1}^{M(s)}\subset\mathbf{N}$ with $p_\mu(s)\ge J$ for all μ such that

$$\begin{aligned}\|x_s^J\|_{N_{s-1}^J,0} \le &\max\left(\bigcup_{\mu=1}^{M(s)}\left\{\frac{\theta_{p_\mu(s)}}{\theta_{p_\mu(s)-J}}\|x_t^0\|_{N_{t-1}^0,p_\mu(s)-J}\colon \operatorname{ran}(x_t^0)\subseteq F_\mu(s)\right\}^*\right)\\ &+\sum\{\varepsilon_t^k\colon x_t^k\in\mathcal{T}(x_s^J,J)\}.\end{aligned}$$

If $0<p_\ell'-1\le J$ for some ℓ, and $\operatorname{ran}(x_s^J)\subseteq F_\ell'$ then, by (4),

$$\begin{aligned}\|x_s^J\|_{N_{s-1}^J,p_\ell'-1} \le &\theta_{p_\ell'-1}\max\left\{\|x_t^{J-p_\ell'+1}\|_{N_{t-1}^{J-p_\ell'+1},0}\colon \operatorname{ran}(x_t^{J-p_\ell'+1})\subseteq\operatorname{ran}(x_s^J)\right\}\\ &+\sum\{\varepsilon_t^k\colon x_t^k\in\mathcal{T}(x_s^J,p_\ell'-1)\}.\end{aligned}$$

For the remaining ℓ's we have by (3), for $j=J$, $p=p_\ell'-1$ and $p'=p_\ell'-1-J$,

$$\begin{aligned}\|x_s^J\|_{N_{s-1}^J,p_\ell'-1} \le &\frac{\theta_{p_\ell'-1}}{\theta_{p_\ell'-1-J}}\max\{\|x_t^0\|_{N_{t-1}^0,p_\ell'-1-J}\colon \operatorname{ran}(x_t^0)\subseteq\operatorname{ran}(x_s^J)\}\\ &+\sum\{\varepsilon_t^k\colon x_t^k\in\mathcal{T}(x_s^J,J)\}.\end{aligned}$$

Combining these estimates we get

$$\|x_i^{J+1}\|_{N_{i-1}^{J+1},0}$$

$$\leq \max\Bigg(\bigcup_{\{\ell:\, p'_\ell=1\}} \bigcup_{\{s:\ \mathrm{ran}(x_s^J)\subseteq F'_\ell\}} \bigcup_{\mu=1}^{M(s)} \Big\{ \frac{\theta_1\theta_{p_\mu(s)}}{\theta_{p_\mu(s)-J}} \|x_t^0\|_{N_{t-1}^0, p_\mu(s)-J}$$

$$+ \sum\{\varepsilon_w^k \colon x_w^k \in \mathcal{T}(x_s^J, J)\} \colon \mathrm{ran}(x_t^0) \subseteq F_\mu(s)\Big\}^*$$

$$\cup \bigcup_{\{\ell:\, 0<p'_\ell-1\leq J\}} \{\theta_{p'_\ell} \|x_t^{J-p'_\ell+1}\|_{N_{t-1}^{J-p'_\ell+1},0}$$

$$+ \sum\{\varepsilon_s^k \colon x_s^k \in \mathcal{T}^*(x_i^{J+1}, p'_\ell)\} \colon \mathrm{ran}(x_t^{J-p'_\ell+1}) \subseteq F'_\ell\}^*$$

$$\cup \bigcup_{\{\ell:\, p'_\ell>J+1\}} \{\frac{\theta_{p'_\ell}}{\theta_{p'_\ell-(J+1)}} \|x_t^0\|_{N_{t-1}^0, p'_\ell-(J+1)}$$

$$+ \sum\{\varepsilon_s^k \colon x_s^k \in \mathcal{T}^*(x_i^{J+1}, J+1)\} \colon \mathrm{ran}(x_t^0) \subseteq F'_\ell\}^* \Bigg) + \varepsilon_i^{J+1}.$$

The induction hypothesis gives that for $0 < p'_\ell - 1 < J$ and $1 \leq t \leq N^{J-p'_\ell+1}$ with $\mathrm{ran}(x_t^{J-p'_\ell+1}) \subseteq F'_\ell$, there exist $K(\ell,t) \in \mathbf{N}$ and sets $G_1(\ell,t) < G_2(\ell,t) < \cdots < G_{K(\ell,t)}(\ell,t)$ which don't split the x_s^0's such that $\bigcup_k G_k(\ell,t) \subseteq \mathrm{ran}(x_t^{J-p'_\ell+1})$, and there exist $(q_k(\ell,t))_{k=1}^{K(\ell,t)} \subset \mathbf{N}$ with $q_k(\ell,t) \geq J - p'_\ell + 1$ such that

$$\|x_t^{J-p'_\ell+1}\|_{N_{t-1}^{J-p'_\ell+1},0} \leq$$

$$\max\Bigg(\bigcup_{k=1}^{K(\ell,t)} \Big\{ \frac{\theta_{q_k(\ell,t)}}{\theta_{q_k(\ell,t)-(J-p'_\ell+1)}} \|x_s^0\|_{N_{s-1}^0, q_k(\ell,t)-(J-p'_\ell+1)} \colon \mathrm{ran}(x_s^0) \subseteq G_k(\ell,t)\Big\}^* \Bigg)$$

$$+ \sum\{\varepsilon_s^k \colon x_s^k \in \mathcal{S}(\ell,t)\}$$

where $\mathcal{S}(\ell,t) = \mathcal{T}(x_t^{J-p'_\ell+1}, J - p_\ell + 1)$. Thus, these estimates give

$$\|x_i^{J+1}\|_{N_{i-1}^{J+1},0}$$

$$\leq \max\Bigg(\bigcup_{\{\ell:\, p'_\ell=1\}} \bigcup_{\{s:\ \mathrm{ran}(x_s^J)\subseteq F'_\ell\}} \bigcup_{\mu=1}^{M(s)} \Big\{ \frac{\theta_1\theta_{p_\mu(s)}}{\theta_{(1+p_\mu(s))-(J+1)}} \|x_t^0\|_{N_{t-1}^0, p_\mu(s)-J}$$

$$+ \sum\{\varepsilon_w^k \colon x_w^k \in \mathcal{T}(x_s^J, J)\} \colon \mathrm{ran}(x_t^0) \subseteq F_\mu(s)\Big\}^*$$

$$\cup \bigcup_{\{\ell:\, 0<p'_\ell-1<J\}} \bigcup_{k=1}^{K(\ell,t)} \Big\{ \frac{\theta_{p'_\ell}\theta_{q_k(\ell,t)}}{\theta_{(p'_\ell+q_k(\ell,t))-(J+1)}} \|x_s^0\|_{N^0_{s-1},q_k(\ell,t)-(J-p'_\ell+1)}$$

$$+\sum\{\varepsilon_s^k\colon x_s^k \in \mathcal{T}^*(x_i^{J+1},p'_\ell)\cup \mathcal{S}(\ell,t)\}\colon \operatorname{ran}(x_s^0)\subseteq G_k(\ell,t)\Big\}^*$$

$$\cup \bigcup_{\{\ell:\, p'_\ell=J+1\}} \{\theta_{J+1}\|x_t^0\|_{N^0_{t-1},0}$$

$$+\sum\{\varepsilon_s^k\colon x_s^k \in \mathcal{T}^*(x_i^{J+1},J+1)\}\colon \operatorname{ran}(x_t^0)\subseteq F'_\ell\}^*$$

$$\cup \bigcup_{\{\ell:\, p'_\ell>J+1\}} \Big\{ \frac{\theta_{p'_\ell}}{\theta_{p'_\ell-(J+1)}} \|x_t^0\|_{N^0_{t-1},p'_\ell-(J+1)}$$

$$+\sum\{\varepsilon_s^k\colon x_s^k \in \mathcal{T}^*(x_i^{J+1},J+1)\}\colon \operatorname{ran}(x_t^0)\subseteq F'_\ell\Big\}^*\Bigg) + \varepsilon_i^{J+1}.$$

Note that $\theta_1\theta_{p_\mu(s)} \leq \theta_{1+p_\mu(s)}$, $1+p_\mu(s) \geq J+1$, $\theta_{p'_\ell}\theta_{q_k(\ell,t)} \leq \theta_{p'_\ell+q_k(\ell,t)}$, $p'_\ell + q_k(\ell,t) \geq p'_\ell + (J-p'_\ell+1) = J+1$ and the sets $F_\mu(s)$'s, F'_ℓ's, and $G_k(\ell,t)$'s don't split the x_s^0's, and arranged in successive order, give the required sequence $F_1 < \cdots < F_m$. Then $1+p_\mu(s)$'s, $p'_\ell + q_k(\ell,t)$'s, and p'_ℓ's for $p'_\ell \geq J+1$, arranged in the corresponding order, give the required sequence $(p_\ell)_{\ell=1}^m$. This finishes the induction.

(6) Since x_i^J is a $(J-1,(\varepsilon_i^{j+1}),\theta_1,N)$ average of (x_i^1) w.r.t. (e_i), by applying (5′) we obtain that there exist intervals $F_1 < F_2 < \cdots < F_m$ ($\bigcup_\ell F_\ell \subseteq \operatorname{ran}(x_i^J)$) which don't split the x_s^1's and $(p_\ell)_{\ell=1}^m \subset \mathbf{N}$ with

$$\|x_i^J\|_{N^J_{i-1},0} \leq \max\left(\bigcup_{\ell=1}^m \left\{\frac{\theta_{p_\ell}}{\theta_{p_\ell-(J-1)}}\|x_s^1\|_{N^1_{s-1},p_\ell-(J-1)}\colon \operatorname{ran}(x_s^1)\subseteq F_\ell\right\}^*\right)$$
$$+\sum\{\varepsilon_s^k\colon x_s^k\in\mathcal{T}(x_i^J,J-1)\}.$$

Now by applying Remark 4.6 (3) we obtain that for ℓ and s with $\operatorname{ran}(x_s^1)\subseteq F_\ell$

$$\|x_s^1\|_{N^1_{s-1},p_\ell-(J-1)} \leq 1+\varepsilon_s^1.$$

Thus

$$\|x_i^J\|_{N^J_{i-1},0} \leq \max\left(\bigcup_{\ell=1}^m \left\{\frac{\theta_{p_\ell}}{\theta_{p_\ell-(J-1)}}\colon \operatorname{ran}(x_s^1)\subseteq F_\ell\right\}^*\right)+\varepsilon$$
$$=\left(\bigcup_{\ell=1}^m \left\{\theta^{J-1}\frac{\phi_{p_\ell}}{\phi_{p_\ell-(J-1)}}\colon \operatorname{ran}(x_s^1)\subseteq F_\ell\right\}^*\right)+\varepsilon$$
$$\leq \theta^{J-1}+\varepsilon.$$

(7) By (4) and (6) we obtain

$$\begin{aligned}\|x\|_{N,p} \leq& \theta_p \max \|x_s^{M-p}\|_{N_{s-1}^{M-p},0} \\ \leq& \theta_p \theta^{M-p-1} + \varepsilon \\ =& \phi_p \theta^{M-1} + \varepsilon. \quad ∎\end{aligned}$$

To prove Theorem 4.4 we need also the following:

Lemma 4.12: *For all $J, N \in \mathbf{N}$, $\varepsilon > 0$ and $Y \prec X = T(\theta_i, S_i)_{\mathbf{N}}$ there exists $y \in Y$ with $\|y\| = 1$ and*

$$\|y\|_{N,p} < \frac{\phi_p}{\theta}(1+\varepsilon), \qquad \textit{for all } p = 1, \ldots, J.$$

Proof: The lemma follows immediately from Lemma 4.11 (7) and the following

Claim: *Let (x_i) be a normalized block basis of (e_i) in $T(\theta_i, S_i)$, J and N be natural numbers, $\delta > 0$ and $(\varepsilon_i^j) \subset (0,1)$. There exists a $(J, (\varepsilon_i^j), \theta_1, N)$ average y of (x_i) w.r.t. such that $\|y\| \geq (1-\delta)\theta^J$.*

Suppose the claim were false. Construct a block sequence (y_i^1) of (x_i) where each y_i^1 is a $(J, (\varepsilon_i^j), \theta_1, N)$ average of (x_i) w.r.t. (e_i). Note that $\|y_i^1\| \leq (1-\delta)\theta^J$. Set $z_i^1 = y_i^1/\|y_i^1\|$ for all $i \in \mathbf{N}$. Let (y_i^2) be a block sequence of $(J, (\varepsilon_i^j), \theta_1, N)$ averages of (z_i^1) w.r.t. (e_i). Note that $\|y_i^2\| \leq (1-\delta)\theta^J$. If $y_i^2 = \sum_{k \in F_i^2} a_k x_k$ then by Remark 4.6 (1) we have that $\sum_{k \in F_i^2} a_k \geq (1-\delta)^{-1}\theta^{-J}$. Set $z_i^2 = y_i^2/\|y_i^2\|$ and continue in the same manner. After m steps we get a vector y^m which is a $(J, (\varepsilon_i^j), \theta_1, N)$ average of (z_i^{m-1}) w.r.t. (e_i). Moreover, writing y^m in the form $y^m = \sum_{k \in F^m} a_k x_k$ we get $\sum_{k \in F^m} a_k \geq (1-\delta)^{-(m-1)}\theta^{-(m-1)J}$. The family $(x_k)_{k \in F^m}$ is mJ-admissible so we have a general estimate

$$\|y^m\| \geq \theta_{mJ} \sum_{k \in F^m} a_k \geq \theta_{mJ}(1-\delta)^{-(m-1)}\theta^{-(m-1)J}.$$

Combining this with an upper estimate for the norm of y^m we get

$$\theta_{mJ}(1-\delta)^{-(m-1)}\theta^{-(m-1)J} \leq (1-\delta)\theta^J.$$

Thus $\phi_{mJ} \leq (1-\delta)^m$. But if m is sufficiently large, this contradicts the definition of θ and this completes the proof of the claim. ∎

Note that the above proof yields that the vector y which satisfies the statement of Lemma 4.12 is a multiple of some $(J, (\varepsilon_i^j), \theta_1, N)$ average of some normalized block sequence of Y w.r.t. (e_i).

Proof of Theorem 4.4: Let $\varepsilon > 0$ be arbitrary. By Lemma 4.12 we can find a normalized block sequence (x_i) in Y and an increasing sequence $(\bar{j}_i)$ of integers, $\bar{j}_1 = 1$, so that if $N_0 = 1$ and $N_i = \max(\text{ran}(x_i))$ w.r.t. (e_s) then for every $i \in \mathbf{N}$ we have

$$\begin{aligned} &\forall p = 1, \ldots, \bar{j}_i, \quad \|x_i\|_{N_{i-1},p} < \frac{\phi_p}{\theta}(1+\varepsilon) \quad \text{and} \\ &\forall p \geq \bar{j}_{i+1}, \qquad \|x_i\|_{N_{i-1},p} < \varepsilon. \end{aligned}$$

Let $(\varepsilon_i^k)_{i,k\in\mathbf{N}} \subset (0,1)$ with $\sum_{i,k} \varepsilon_i^k < \varepsilon$ and let x be a $(j, (\varepsilon_i^k), \theta_1, 1)$ average of (x_i) w.r.t. (e_i) with admissible averaging tree $(x_i^k)_{k=0,i=1}^{j,N^k}$ of (x_i) and maximum coordinates $(N_i^k)_{k=1,i=0}^{j,N^k}$ w.r.t. (e_i). For $i = 1, \ldots, N^0$ if $x_i^0 = x_s$, define $j_i = \bar{j}_s$. Then $j_1 < \cdots < j_{N^0}$ and for $i = 1, \ldots, N^0$ we have

$$\begin{aligned} &\forall p = 1, \ldots, j_i, \quad \|x_i^0\|_{N_{i-1}^0,p} < \frac{\phi_p}{\theta}(1+\varepsilon) \quad \text{and} \\ &\forall p \geq j_{i+1}, \qquad \|x_i^0\|_{N_{i-1}^0,p} < \varepsilon. \end{aligned}$$

Note (by Remark 4.6 (2)) that x_1^j is j-admissible w.r.t. (x_i) and by Lemma 4.11 (5) there exist $m \in \mathbf{N}$, intervals $F_1 < \cdots < F_m$ which don't split the x_s^0's, and $(p_\ell)_{\ell=1}^m \subset \mathbf{N}$ with $p_\ell \geq j$ for all ℓ such that

$$\|x\| \leq \max\left(\bigcup_{\ell=1}^m \left\{\frac{\theta_{p_\ell}}{\theta_{p_\ell - j}} \|x_s^0\|_{N_{s-1}^0, p_\ell - j} \colon \text{ran}(x_s^0) \subseteq F_\ell\right\}^*\right) + \varepsilon.$$

For each $\ell = 1, \ldots, m$ if $p_\ell > j$ then there exists exactly one $m_\ell \in \mathbf{N}$ such that $j_{m_\ell} \leq p_\ell - j < j_{m_\ell+1}$. We shall use the obvious remark that if $A \subseteq [0,\infty)$ is a finite non-empty set and $a \in A$ then $\max(A^*) \leq \max(A \backslash \{a\})$. If $p_\ell = j$ then $\theta_{p_\ell}/\theta_{p_\ell - j} = \theta_j$ and we note that

$$\|x_s^0\|_{N_{s-1}^0,0} \leq \frac{1}{\theta_1}\|x_s^0\| = \frac{1}{\theta_1}.$$

Thus

$$\|x\| \leq \max\left(\bigcup_{\{\ell\colon p_\ell > j\}} \left\{\frac{\theta_{p_\ell}}{\theta_{p_\ell - j}} \|x_s^0\|_{N_{s-1}^0, p_\ell - j} \colon \text{ran}(x_s^0) \subseteq F_\ell, s \neq m_\ell\right\} \cup \left\{\frac{\theta_j}{\theta_1}\right\}\right) + \varepsilon.$$

Let $\text{ran}(x_s^0) \subseteq F_\ell$ and $p_\ell > j$. If $s < m_\ell$ we have $j_{s+1} \leq j_{m_\ell} \leq p_\ell - j$ and so $\|x_s^0\|_{N_{s-1}^0, p_\ell - j} < \varepsilon$. If $s > m_\ell$ we have $j_s \geq j_{m_\ell+1} > p_\ell - j$ and so

$$\|x_s^0\|_{N_{s-1}^0, p_\ell - j} < \frac{\phi_{p_\ell - j}}{\theta}(1+\varepsilon).$$

Note that

$$\frac{\theta_{p_\ell}\phi_{p_\ell - j}}{\theta_{p_\ell - j}\theta} = \theta^{j-1}\phi_{p_\ell}$$

and therefore

$$\|x\| \le \theta^{j-1}\sup_{p\ge j}\phi_p(1+\varepsilon)\vee\frac{\theta_j}{\theta_1}+2\varepsilon.$$

Note (by Remark 4.6) that we can write $x=\sum_F a_i x_i$ for some set $F\in S_j$ where $a_i>0$ for all $i\in F$ and $\sum_{i\in F}a_i=1$. Therefore $\delta_j(Y)\le\|x\|$ and since $\varepsilon>0$ is arbitrary we obtain the result. ∎

Note that Theorem 4.4 does not necessarily give the best possible estimate for $\delta_j(Y)$. Indeed if $\theta_n=2^{-n}$ for all n then $T=T(\theta_n,S_n)_{\mathbf N}$ is Tsirelson's space and, for all $Y\prec T$, $\delta_j(Y)=2^{-j}$ [OTW]. Yet Theorem 4.4 only gives $\delta_j(Y)\le 2^{-j+1}$. However, we have the following estimate which does yield the proper estimate for Tsirelson's space.

THEOREM 4.13: *Let $X=T(\theta_n,S_n)_{\mathbf N}$ be regular. Then for all $Y\prec X$ and $j\in\mathbf N$ we have*

$$\delta_j(Y)\le\theta^j\sup_{p\ge j}\frac{\phi_p}{\phi_{p-j}}.$$

Proof: Let $Y\prec X$, $j\in\mathbf N$ and $\varepsilon>0$. Since Y contains ℓ_1^n's uniformly, for all $N\in\mathbf N$ $\exists y\in Y$ with $1=\|y\|\le\|y\|_{N,0}\le 1+\varepsilon$ (see e.g. [OTW] proposition 2.7). Therefore we may choose inductively a normalized block sequence (x_i) in Y so that for $i\in\mathbf N$, if $N_i=\max(\mathrm{ran}(x_i^0))$ w.r.t. (e_i) ($N_0=1$) then $\|x_i\|_{N_{i-1},0}\le 1+\varepsilon$. Note then that for every $i,p\in\mathbf N$, $\|x_i\|_{N_{i-1},p}\le\|x_i\|_{N_{i-1},0}\le 1+\varepsilon$. Let $(\varepsilon_i^k)_{i,k\in\mathbf N}\subset(0,1)$ with $\sum_{i,k}\varepsilon_i^k<\varepsilon$ and let x be a $(j,(\varepsilon_i^k),\theta_1,1)$ average of (x_i) w.r.t. (e_i) with admissible averaging tree $(x_i^k)_{k=0,i=1}^{j,N^k}$ of (x_i) and maximum coordinates $(N_i^k)_{k=0,i=0}^{j,N^k}$ w.r.t. (e_i). Note then that for every $i,p\in\mathbf N$ we have that $\|x_i^0\|_{N_{i-1}^0,p}\le 1+\varepsilon$. By Lemma 4.11 (5) there exist $m\in\mathbf N$, $F_1<\cdots<F_m$ intervals in $\mathbf N$ which don't split the x_i^0's and integers $(p_\ell)_{\ell=1}^m$ with $p_\ell\ge j$ for all ℓ, such that

$$\begin{aligned}\|x\|\le&\max\left\{\frac{\theta_{p_\ell}}{\theta_{p_\ell-j}}\|x_i^0\|_{N_{i-1}^0,p_\ell-j}\colon \ell=1,\dots,m,\mathrm{ran}(x_i^0)\subseteq F_\ell\right\}+\varepsilon\\ \le&\theta^j\sup_{p\ge j}\frac{\phi_p}{\phi_{p-j}}(1+\varepsilon)+\varepsilon\end{aligned}$$

and the result follows since $\varepsilon>0$ is arbitrary. ∎

To estimate $\delta_j(Y)$ for $Y=X$ is easy as we see from the following:

THEOREM 4.14: *Let $X = T(\theta_n, S_n)_{n\in\mathbf{N}}$ be regular. Then for all $j \in \mathbf{N}$ we have $\delta_j(X) = \theta_j$.*

Proof: Let $j \in \mathbf{N}$ and $\varepsilon > 0$. Let $(\varepsilon_i^k)_{i,k\in\mathbf{N}} \subset (0,1)$ with $\sum \varepsilon_i^k < \varepsilon$ and let x be a $(j, (\varepsilon_i^k), \theta_1, 1)$ average of (e_i) w.r.t. (e_i) with admissible averaging tree $(x_i^k)_{k=0,i=1}^{j,N^k}$ and maximum coordinates $(N_i^k)_{k=0,i=0}^{j,N^k}$ w.r.t. (e_i). Then by (2) there exists $m \in \mathbf{N}$, $F_1 < \cdots < F_m$ intervals in $\mathbf{N}$ and integers $(p_\ell)_{\ell=1}^m$ with $p_\ell \geq j$ for all ℓ, such that

$$\|x\| \leq \max\left\{ \frac{\theta_{p_\ell}}{\theta_{p_\ell - j}} \|x_i^0\|_{N_{i-1}^0, p_\ell - j} \colon \ell = 1, \ldots, m, \operatorname{ran}(x_i^0) \subseteq F_\ell \right\} + \varepsilon.$$

Since $(x_i^0)_{i=1}^{N^0}$ is a subsequence of (e_i), we have $\|x_i^0\|_{N_{i-1}^0, p_\ell - j} = \theta_{p_\ell - j}$ for every $i = 1, \ldots, N^0$ and $\ell = 1, \ldots, m$. Thus $\|x\| \leq \max_{1 \leq \ell \leq m} \theta_{p_\ell} + \varepsilon$. Since the sequence (θ_i) is decreasing we have $\|x\| \leq \theta_j + \varepsilon$. Since $\operatorname{supp}(x) \in S_j$ and $\varepsilon > 0$ is arbitrary we obtain the result. ∎

QUESTION: *If $X = T(\theta_n, S_n)_{\mathbf{N}}$ is a regular mixed Tsirelson space and $Y \prec X$, is $\delta_j(Y) = \theta_j$ for every $j \in \mathbf{N}$?*

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